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Ward-Type Identities for the Two-Dimensional Anderson Model at Weak Disorder

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Using the particular momentum conservation laws in dimension $d=2$, we rewrite the Anderson model in terms of low-momentum long-range fields, at the price of introducing electron loops. The corresponding loops satisfy a Ward-type identity, hence are much smaller than expected. This fact should be useful for the study of the weak-coupling model in the middle of the spectrum of the free Hamiltonian.

KEY WORDS:

1. INTRODUCTION

We consider a continuous Anderson model in dimension $d=2$. The model is defined through the following Hamiltonian

$$H=-\Delta+\lambda V \tag{1}$$

where V is a Gaussian random field which is a regularized white noise.

We are interested in the density of states at weak disorder ($\lambda \ll 1$) and in the free spectrum (at energy $E>0$). It is well known[1] that because of ergodicity, the density of states is a deterministic quantity given by

$$\rho(E)=\frac{1}{\pi} \lim_{\sigma \to 0} \lim_{\Lambda \to \infty} \mathbb{E}[\operatorname{Im} G(E+i\sigma; 0, 0)] \tag{2}$$

where G is the resolvent, or Green's function, of the system

$$G(z)=(H-z)^{-1} \tag{3}$$

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The limit $\Lambda \to \infty$ stands for the fact that we must work in a finite volume to have well defined quantities and then take the thermodynamic limit.

Thus the problem amounts to studying the mean Green's function. Perturbations suggest that

$$\mathbb{E}[G(E+i\sigma)] \sim \frac{1}{p^2 - E - i\sigma - \Sigma} \tag{4}$$

where the self-energy Σ is given at leading order by the Born approximation,[2, 3]

$$C = \frac{1}{p^2 - E - i\sigma - \Sigma_{\text{Born}}} \tag{5}$$

$$\Sigma_{\text{Born}} = \lambda^2 \mathbb{E}(VCV) \tag{6}$$

which yields a finite imaginary part of order λ^2.

In this paper, we derive a Ward type identity which should allow to control the mean Green's function for *initial* imaginary part of order $\lambda^{2+\varepsilon}$, i.e., much smaller than the expected final imaginary part. This is not enough to go to the limit $\sigma \to 0$ which is in fact equivalent to $\sigma \simeq \lambda^3$ thanks to spectral averaging techniques[4] or equivalently complex translation of the potential.[5] Nevertheless, we think that this kind of identity should play a role in studying the mean Green's function of the model and in proving its expected long range decay.

We give first a heuristic presentation of this identity which is a little bit complicated. Then we will derive it in a simplified model in a single cube of size $\lambda^{-2-\varepsilon}$. This result, when combined with a polymer expansion of the resolvent would allow to control the thermodynamic limit of the model with $\sigma = \lambda^{2+\varepsilon}$ in the same way than refs. 5 and 6.

2. PHASE SPACE PICTURE AND MATRIX MODEL

Our study is based on a phase space multiscale analysis.[5, 7, 6] We divide the momentum space into slices such that in the jth slice Σ_j, we have $M^{-j-1} \leqslant |p^2 - E| \leqslant M^{-j}$ for some integer $M \geqslant 2$. Then the real space is divided into lattices $\mathbb{D}_j$ of cubes of dual size M^j.

The point is that the potential V seen as an operator $\mathbf{V}$ has a kernel in momentum space given by

$$\mathbf{V}(p, q) = \hat{V}(p - q) \tag{7}$$

When p and q are restricted to low slices, i.e. very close to $p^2 = E$, knowing the momentum transfer $p - q$ allows to recover back the pair $\{p, -q\}$[8, 9] so that the potential has a very strong matrix flavor.[7]

We call η_j a smoothed projector on the slice Σ_j that we further divide into angular sectors S^j_α of width $M^{-j/2}$, corresponding to some n_{α_j}. We write $\bar{\alpha}$ for the opposite sector to α and we introduce also the notation

$$\eta_j = \sum_{k \geqslant j} \eta_k \tag{8}$$

For any operator A we write

$$A^{jk} = \eta_j A \eta_k \tag{9}$$

Finally, for any lattice $\mathbb{D}_j$ of cubes Δ, we can make an orthogonal decomposition of the field V into a sum of fields V_Δ, the support of V_Δ being on a close neighborhood of Δ.[5] Then, using the matrix aspect of the potential, we can derive the following estimates.[5, 6]

Lemma 1. There are constants K_1 and K_2 such that for all $j \leqslant k$, $a \geqslant 1$ and $\Delta \in \mathbb{D}_k$

$$\mathbb{P}(\| V^{j\bar{k}}_\Delta \| \geqslant aK_1 M^{-j/2}) \leqslant K_2 e^{-a^2 M^{k/2 - j/3}} \tag{10}$$

Lemma 2 (Tadpole-free operators). There are constants K_1 and K_2 such that for all $j \leqslant k$, $a \geqslant 1$ and $\Delta, \Delta' \in \mathbb{D}_k$

$$\mathbb{P}(\| :V^{\bar{k}j}_\Delta C V^{j\bar{k}}_{\Delta'}: \| \geqslant aK_1 M^{-(k-j)/2}) \leqslant K_2 e^{-a^2 M^{j/6}} \tag{11}$$

where : : stands for the Wick ordering

$$:V^{\bar{k}j}_\Delta C V^{j\bar{k}}_{\Delta'}: = V^{\bar{k}j}_\Delta C V^{j\bar{k}}_{\Delta'} - \langle V^{\bar{k}j}_\Delta C V^{j\bar{k}}_{\Delta'} \rangle \tag{12}$$

Lemma 3 (Almost diagonal operators). There are constants K_1 and K_2 such that for all $j \leqslant k$, $0 < r < 1$, $a \geqslant 1$ and $\Delta \in \mathbb{D}_k$

$$\mathbb{P}(\| {}^{(r)}\underline{V}^{j\bar{k}}_\Delta \| \geqslant aK_1 M^{-(j/2)(1 + r/2)}) \leqslant K_2 e^{-a^2 M^{k/2 - j/3 - rj/6}} \tag{13}$$

where

$${}^{(r)}V^{jk}_\Delta \equiv \sum_{|\alpha - \beta| \leqslant M^{-rj/2}} \eta_{\alpha_j} V_\Delta \eta_{\beta_k} + \sum_{|\alpha - \bar{\beta}| \leqslant M^{-rj/2}} \eta_{\alpha_j} V_\Delta \eta_{\beta_k} \tag{14}$$

3. RESULT

We are looking at a simplified model in a single cube: in $\mathbb{R}^2/\lambda^{-2-\varepsilon}\mathbb{Z}^2$ we consider the following Hamiltonian

$$H = -\Delta + \lambda V \tag{15}$$

where V is a Gaussian random field with translation invariant covariance $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $\xi^{1/2} \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ (the square root being taken in operator sense). The result could be extended to $\xi \in \mathscr{S}(\mathbb{R}^2)$ but the development would be much heavier.

We are interested in computing the mean Green's function

$$\bar{G}(E - i\sigma) = \int d\mu(V)(H - E - i\sigma)^{-1} \tag{16}$$

for which we have the following result

Theorem 1 (Ward type identity). There exists $\nu > 0$ such that for all $\sigma \leqslant \lambda^2$

$$\|\bar{G}(E) - C\| \leqslant O(1)\left(\frac{\lambda^{2+\nu}}{\sigma}\right)^3 \frac{1}{\sigma} \tag{17}$$

where C is the renormalized propagator at leading order (Born approximation).

Moreover, we can iterate the development in order to obtain an asymptotic expansion to all orders.

This result together with a polymer expansion[5] would allow to control the thermodynamic limit for imaginary part

$$\sigma \geqslant \lambda^{2+\varepsilon} \tag{18}$$

Therefore we can investigate the mean Green's function up to a region with imaginary part much smaller than the expected final one. We cannot for the moment go to the real axis which would require to have $\nu > 1$. Nevertheless, we think that this kind of identity should play a role in the study of the mean Green's function and of the density of states.

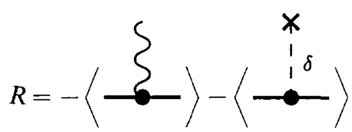
4. HEURISTIC PRESENTATION

Let us assume that we are working in the region of momenta $|p^2 - E| \leqslant M^{-j_1}$, where

$$j_1 = (1 - \nu_1)\, j_0 \tag{19}$$

$$\lambda^2 (\log \lambda^{-1})^2 \simeq M^{-j_0} \tag{20}$$

We are looking at the off-diagonal part of the potential, i.e., the $V_{\alpha\beta}$'s such that α is quite different from β or $\bar{\beta} \equiv \pi + \beta$. We know that in this case, sectors are preserved up to $M^{-j_0/2}$ and even better.[6] If we introduce a counter-term δ and perform one step of perturbation, e.g., by putting an interpolation parameter on the potential and the counter-term, we get a remainder term which looks like



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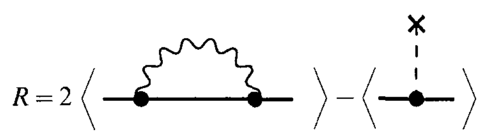
We can integrate the V by parts and get



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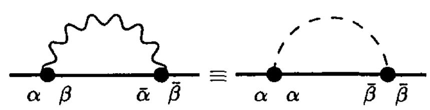
Sector conservation tells us that there are two possible configurations:



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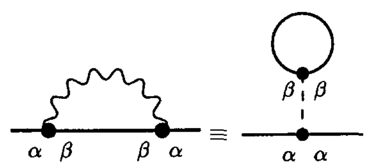

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We used the fact that V being almost ultra-local we can identify both ends of its propagator (the wavy line) and replace it by the dashed line which corresponds to the low momentum channel. Furthermore, for the first term, we used

$$(n_\beta G \eta_{\bar{\alpha}})(x, x) = (\eta_\beta G \eta_{\bar{\alpha}})^{t}\,(x, x) = (\eta_{\alpha} G \eta_{\beta})(x, x) \tag{25}$$

where A^{t} stands for the transposed operator, whose kernel is

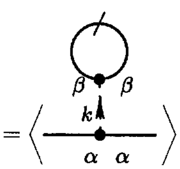
$$A^{\mathrm{t}}(x, y) = A(y, x) \tag{26}$$

The first configuration corresponds to the insertion of two fields of very low momentum, i.e., to almost diagonal operators, and as such is very small. Thus we just need to see how the second term, which is the insertion of a tadpole, will kill the counter-term, at least at leading order.

First, we can remark that if α and β are far enough from each other, the momentum flowing into the loop has a size M^{-j_1}, being at the intersection of two tubes of size $M^{-j_0/2} \times M^{-j_1}$. But we can go further: if all the incoming legs at the vertex are in the "very low" slice Σ_{j_0}, the ingoing momentum has a size M^{-j_0}. This means that either we have a very small momentum flowing into the loop or one of the four legs is "high," which means that $|p^2 - E|$ is large. This allows to earn a small factor.

If we set the counter-term equal to the tadpole with the bare propagator C_0, we are led to study

$\Gamma_{\alpha\beta}(k) =$



(27)

where the slashed line stands for $G - C_0$. In momentum space the contribution of the loop is

$$\int dp\, \eta_\beta(p)\, \eta_\beta(p+k)(G - C_0)(p+k, p) \tag{28}$$

The key point is then to notice that when p is close to k_β, the center of the sector β in momentum space, one can write

$$2k.k_\beta + k^2 = [(p+k)^2 - E - i\sigma] - [p^2 - E - i\sigma] - 2k.(p - k_\beta) \tag{29}$$

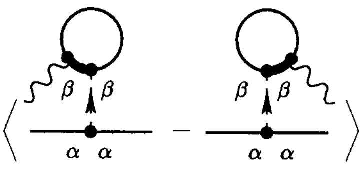
$$= C_0^{-1}(p+k) - C_0^{-1}(p) + O(|k|\, |p - k_\beta|) \tag{30}$$

Using the resolvent identity

$$G - C_0 = -C_0 \lambda V G = -G \lambda V C_0 \tag{31}$$

we get

$$(2k.k_\beta + k^2)\, \Gamma_{\alpha\beta}(k) \simeq$$



(32)

where the thick line stands for η_β.

The perturbative reformulation of this identity is the following. Starting from a vertex function

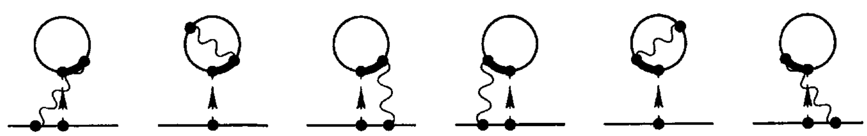
$$\Gamma^{(\beta)}_{2,1}(p, k) = C_\beta(p)\, C_\beta(p+k) \tag{33}$$

we have the identity

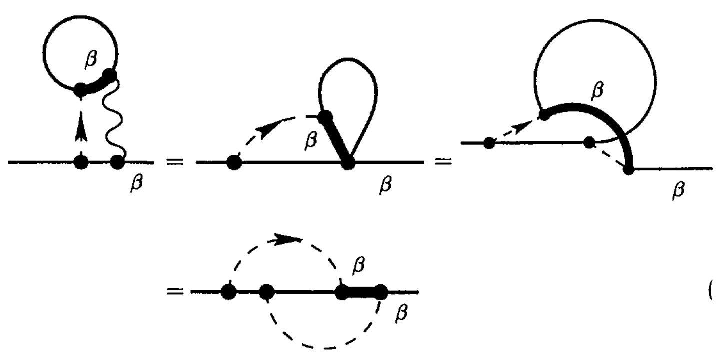
$$(2k.k_\beta + k^2)\, \Gamma^{(\beta)}_{(2,1)}(p, k) \simeq C_\beta(p) - C_\beta(p+k) \tag{34}$$

This looks very much like a perturbative Ward identity in quantum electrodynamics where we express the vertex function $\Gamma_{2,1}$ (between an electron, a positron and a photon) in terms of the difference of two electron propagators.

Once again, we integrate by parts the V which has been taken down. This leads to six possible graphs



Then we use sector conservation and perform various unfolding operations. In order to illustrate the process, let us show how it works on the following typical term (using the fact that a wavy line is almost a δ function)



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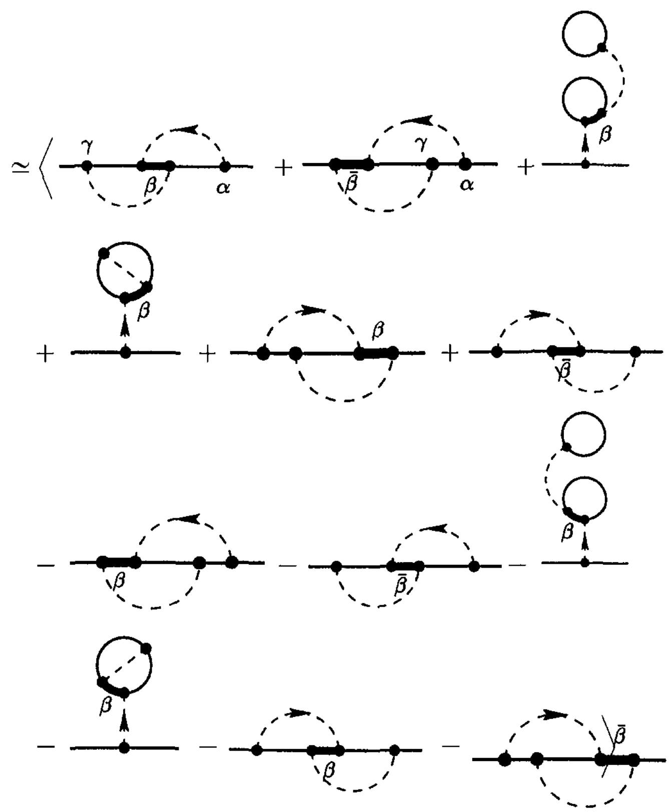
In the end we obtain 12 terms which are

$(2k.k_\beta + k^2)\,\Gamma_{\alpha\beta}(k)$



(36)

We have two types of graphs:

- graphs without electron loops, containing very low momentum field insertions,
- and graphs with remaining electron loops.

The graphs in the first category are small. In the second category, the 3rd, 4th, 9th and 10th graphs form two pairs which almost compensate each other (3–9 and 4–10).

The point is that when all the incoming legs at the various vertices are "very low," the momenta flowing into the loops have size M^{-j_0}. This implies that the dashed lines stand for *propagators* which decay on a length scale M^{j_0}. But η_β (the thick line) decays on a scale $M^{j_1} \ll M^{j_0}$, therefore it is almost-local with respect to the scale M^{j_0} and we can approximate it by a point. This means that the graphs number 3 and 9 as well as the number 4 and 10 compensate each other, up to a gradient term in $M^{-(j_0-j_1)}$.

This conclude our heuristic description of the reasons for which Theorem 1 holds. The next section is devoted to the proof.

5. PROOF OF THEOREM 1

5.1. Notations

First of all, we introduce various notations

- We define $\downarrow \equiv \eta_\downarrow = \eta_{j_1}$ and $\uparrow \equiv \eta_\uparrow = (1-\eta_{j_1})$, so that every operator A can be written

$$A = {}_\uparrow A_\uparrow + {}_\uparrow A_\downarrow + {}_\downarrow A_\uparrow + {}_\downarrow A_\downarrow \tag{37}$$

- for $0<r<1$ to be fixed later, we define

$$\alpha \sim \beta \Leftrightarrow |\alpha-\beta| \leqslant M^{-r j_0/2} \tag{38}$$

- for $K = O(1)$ we note

$$\alpha \simeq \beta \Leftrightarrow |\alpha-\beta| \leqslant K M^{-j_0/2} \tag{39}$$

- finally we introduce

$$j_2 = (1-\nu_2)\, j_0 > j_1 \tag{40}$$

5.2. Starting the Development

We define the counter-term through the self-consistent equation

$$\Sigma(x-y) = \lambda^2 \xi(x-y)\, C(x-y) \tag{41}$$

$$C = \frac{1}{p^2 - E - i\sigma - \Sigma} \tag{42}$$

Then we set

$$G(t) = \frac{1}{p^2 - E - i\sigma - \Sigma + t\lambda V + t^2 \Sigma} \tag{43}$$

$$\bar{G}(t) = \mathbb{E}[G(t)] \tag{44}$$

so that we can write

$$\bar{G}=\bar{G}(1)=\bar{G}(0)+\int_0^1 \partial_t \bar{G}(t)\,dt = C+\int_0^1 \partial_t \bar{G}(t)\,dt \tag{45}$$

Then our problem reduces to the study of

$$\partial_t \bar{G} = -\langle G(\lambda V+2t\Sigma)\,G\rangle \tag{46}$$

$$=2t\int dx\,dy \langle G(.,x)[\lambda^2\xi(x,y)\,G(x,y)-\Sigma(x,y)]\,G(y,.)\rangle \tag{47}$$

$$=2t\int dx\,dy \langle G(.,x)[\lambda^2\xi(x,y)(G-C)(x,y)]\,G(y,.)\rangle \tag{48}$$

$$\equiv 2t\langle G[\lambda^2\xi * (G-C)]\,G\rangle \tag{49}$$

where we have integrated the V by parts from (46) to (47).

If we plug in the "high" and "low" slices we get

$$\partial_t \bar{G} = \sum_{i_1,\dots,i_4\in\{\downarrow,\uparrow\}} 2t\langle G\eta_{i_1}[\lambda^2\xi * \eta_{i_2}(G-C)\,\eta_{i_3}]\,\eta_{i_4} G\rangle \tag{50}$$

$$= \sum_{i_1,\dots,i_4\in\{\downarrow,\uparrow\}} \partial_t \bar{G}_{i_1,\dots,i_4} \tag{51}$$

5.3. Higher Part

We can quite easily deal with the case $(i_1,\dots,i_4)\neq(\downarrow,\dots,\downarrow)$ because we have a high leg.

Lemma 4.

$$\sum_{(i_1,\dots,i_4)\neq(\downarrow,\dots,\downarrow)} \|\partial_t \bar{G}_{i_1,\dots,i_4}\| \leqslant O(1)\,\lambda^{\nu_1}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{52}$$

Proof. We start with

$$G(t)-C=-C(\lambda V+t^2\Sigma)\,G(t) \tag{53}$$

then, we write the covariance ξ as the integration of two insertions of an auxiliary field U.

$$\partial_t \bar{G}_{i_1,\dots,i_4} = -2t\int G\eta_{i_1}(\lambda U)\,\eta_{i_2} C(\lambda V+t^2\Sigma)\,G\eta_{i_3}(\lambda U)\,\eta_{i_4} G\,d\mu(V)\,d\mu(U) \tag{54}$$

At this point we perform a large field versus small field decomposition[5, 6]

$$1 = \varepsilon(U, V) + (1-\varepsilon)(U, V) \tag{55}$$

where ε is a smooth function which forces

$$\| V^{j\bar{k}} \| \leqslant O(1)\, M^{-j_1/2} M^{\tau_1 \max((j_1 - k)/2,\, 0)} \tag{56}$$

$$\| U^{j\bar{k}} \| \leqslant O(1)\, M^{-j_1/2} M^{\tau_1 \max((j_1 - k)/2,\, 0)} \tag{57}$$

$$\| U^{\bar{k}j} C^j V^{j\bar{k}} \| \leqslant O(1)\, M^{-((k-j)/2)} M^{\tau_1 \max[(j_1 - j),\, 0]} \tag{58}$$

The last condition is possible because U and V cannot contract together, therefore we can use Lemma 2 on tadpole-free operators. Thanks to Lemmas 1 and 2, we find that the large field contribution will be small, of order

$$\mathbb{E}[\, \| \partial_t G_{i_1 \cdots i_4} \| \, (1-\varepsilon)] \leqslant O(1)\, \lambda^{-2} e^{-\kappa \lambda^{-2\tau_1(1-\nu_1)}} \times \frac{1}{\sigma^3} \tag{59}$$

for some constant κ.

In the small field region, we will use the fact that we have a high leg so that in some sense we can make perturbations. Suppose that η_{i_4} is the high leg at scale k_0, the operator $\lambda U^{i_3 i_4}$ will have a size $\lambda M^{-k_0/2} \gg \lambda^2$. But if we perform a resolvent expansion on $\lambda U^{i_3 i_4} G$ we will get $\lambda U^{i_3 i_4} C^{k_0} \lambda V G$ thus we earn an extra factor $\lambda U^{i_3 i_4} C^{k_0}$ whose norm is $\lambda M^{k_0/2} \ll 1$. Then we can iterate the process until either we fall back to a scale of order j_0 or we have earned enough small factors.

Lemma 5 (Stairway expansion):

$$\begin{aligned}
(1-\eta_{\bar{m}})\, G = {} & \sum_{k_0 < m} C^{k_0} [1 - (t^2 \Sigma + \lambda V \eta_{\bar{j}_0})\, G] \\
& - \sum_{\substack{k_0 < m \\ k_1 < j_0}} C^{k_0} \lambda V C^{k_1} [1 - (t^2 \Sigma + \lambda V \eta_{\bar{k}_1})\, G] \\
& + \sum_{\substack{k_0 < m \\ 0 < k_1 < j_0 \\ k_2 < k_1}} C^{k_0} \lambda V C^{k_1} \lambda V C^{k_2} [1 - (t^2 \Sigma + \lambda V \eta_{\bar{k}_2})\, G] - \cdots
\end{aligned} \tag{60}$$

where $C^k = \eta_k C$ and $\eta_{\bar{0}} = 1$.

Of course, we have a similar result for $G(1-\eta_{\bar{m}})$ by expanding to the left.

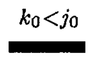
Proof. The proof is by induction on m thanks to the resolvent identity

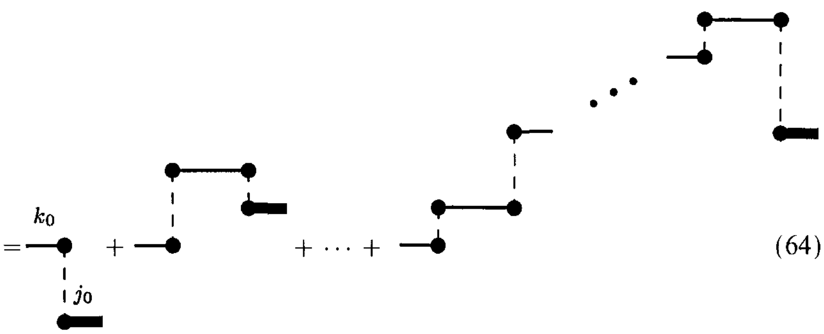
$$G = C - C(t^2\Sigma + \lambda V)\,G \tag{61}$$

that we write as

$$\eta_p G = C^p[1 - C(t^2\Sigma + \lambda V\eta_{\bar{p}})\,G] - \sum_{q<p} C^p \lambda V \eta_q G \quad \blacksquare \tag{62}$$

In Eq. (60) we can group the various terms according to whether they end by a C, a ΣG or a VG. Then we can introduce a diagrammatic representation

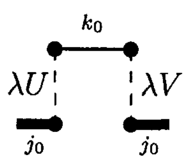
$(1-\eta_{j_0})\,G =$  $= A_V + A_\Sigma + A_C$ (63)

$A_V =$  (64)

A_Σ and A_C can be represented in the same way by changing the rightmost term.

We apply the stairway expansion on each leg which has its momentum above the scale j_0 and is not linked to a long enough stairway. This allows to show that each field insertion behaves like $O(1)\,\lambda^2$. Furthermore, we earn a factor $\lambda M^{j_1/2} M^{\tau_1 j_1/2} \sim \lambda^{\nu_1-\tau_1}$, thanks to the λU which have a high leg. Indeed, the corresponding insertion of λU will transform into the insertion of a sum of stairways. The A_V part has the following form:

- the second order insertion is

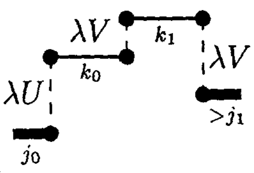
$A_2 \equiv \sum_{k_0<j_1}$  (65)

Since U and V cannot contract together, the small field condition tells us that

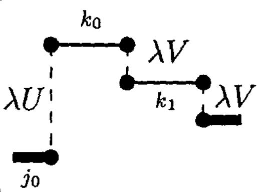
$$\|A_2\| \leqslant O(1) \sum_{k_0<j_1} \lambda^2 M^{-((j_0-k_0)/2)} \leqslant O(1)\, \lambda^2 M^{-((j_0-j_1)/2)} \tag{66}$$

- at third order, we have two possible configurations

$$A_3^{(1)} \equiv \sum_{\substack{k_0<j_1\\ k_1\leqslant k_0}}$$



$$A_3^{(2)} \equiv \sum_{\substack{k_0<j_1\\ k_1> k_0}} \tag{67}$$



For $A_3^{(1)}$, one finds

$$\|A_3^{(1)}\| \lesssim \lambda M^{-k_0/2} M^{k_0}\, \lambda M^{-k_1/2} M^{\tau_1((j_1-k_0)/2)} M^{k_1}\, \lambda M^{-k_1/2} \tag{68}$$

$$\lesssim \lambda^2(\lambda M^{j_1/2})\, M^{-(1-\tau_1)((j_1-k_0)/2)} \tag{69}$$

$$\lesssim \lambda^2(\lambda M^{j_1/2}) \tag{70}$$

In the same way, one finds for $A_3^{(2)}$

$$\|A_3^{(2)}\| \lesssim \lambda^2 M^{-((k_1-k_0)/2)} M^{\tau_1(j_1-k_0)} M^{k_1}\, \lambda M^{-k_1/2} M^{\tau_1((j_1-k_1)/2)} \tag{71}$$

$$\lesssim \lambda^2(\lambda M^{j_1/2})\, M^{-(1-3\tau_1)((j_1-k_0)/2)} \tag{72}$$

$$\lesssim \lambda^2(\lambda M^{j_1/2}) \tag{73}$$

- fourth and higher order terms can be treated similarly. We find the same power counting with more and more small factors as the order increases.

We do the same to bound the A_C and A_Σ parts and get the announced result

$$\|\partial_t \bar{G}_{i_1 \cdots i_4}\| \leqslant O(1)\, \lambda^{n}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \quad \blacksquare \tag{74}$$

5.4. Lower Part

We introduce the angular sectors

$$\partial_t \bar{G}_\downarrow \equiv \partial_t \bar{G}_{\downarrow \cdots \downarrow} = 2t \sum_{\alpha_1 \cdots \alpha_4} \langle G\eta_{\alpha_1}[\lambda^2 \xi * \eta_{\alpha_2}(G-C)\,\eta_{\alpha_3}]\,\eta_{\alpha_4} G\rangle \tag{75}$$

First we extract the *degenerate* part of the sum over the sectors, i.e., the part $(\alpha_1 \sim \alpha_1, \bar{\alpha}_2)$. In order to do so, we define the almost diagonal part of ${}_\downarrow U_\downarrow$ (with momentum close to 0 or $2\sqrt{E}$).

$$_\downarrow U_{\text{diag}\downarrow} = \sum_{\alpha \sim \beta, \bar{\beta}} \eta_\alpha U \eta_\beta \tag{76}$$

Then we write

$$\partial_t \bar{G}_\downarrow = \partial_t \bar{G}_\downarrow^{(0)} + \partial_t \bar{G}_\downarrow^{(1)} \tag{77}$$

$$\partial_t \bar{G}_\downarrow^{(0)} = 2t \int G_\downarrow \lambda U_{\text{diag}\downarrow}(G-C)_\downarrow \lambda U_\downarrow G\, d\mu(U)\, d\mu(V) \tag{78}$$

Keeping only the almost diagonal part for one of the λU allows us to earn a factor $M^{-rj_0/4} = \lambda^{r/2}$ in the small field region, thanks to Lemma 3. Therefore, we can treat $\partial_t \bar{G}_\downarrow^{(0)}$ in the same way we controlled the higher part and get the following bound

$$\|\partial_t \bar{G}_\downarrow^{(0)}\| \leqslant O(1)\, \lambda^{r/2} \left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{79}$$

We are left with

$$\partial_t \bar{G}_\downarrow^{(1)} = 2t\lambda^2 \sum_{\substack{\alpha_1 \nsim \alpha_2, \bar{\alpha}_2 \\ \alpha_3, \alpha_4}} \int dx\, dy\, \xi(x, y)$$

$$\times \langle G\eta_{\alpha_1}(\,.\,, x)\, \eta_{\alpha_2}(x, .)(G-C)\, \eta_{\alpha_3}(\,.\,, y)\, \eta_{\alpha_4}(y, .)\, G\rangle \tag{80}$$

$$= 2t\lambda^2 \sum_{\substack{\alpha_1 \nsim \alpha_2, \bar{\alpha}_2 \\ \alpha_3, \alpha_4}} \int du\, \xi(u) \int dz$$

$$\times \langle G\eta_{\alpha_1}(\,.\,, z)\, \eta_{\alpha_2}(z, .)(G-C)\, \eta_{\alpha_3}(\,.\,, z+u)\, \eta_{\alpha_4}(z+u, .)\, G\rangle \tag{81}$$

In the following, we will note

$$\eta_\gamma^{(u)}(x, y) = \eta_\gamma(x - y + u) = \eta_\gamma(x + u, y) = \eta_\gamma(x, y - u) \tag{82}$$

5.5. Sector Conservation and Unfolding

If we look at the vertex in momentum space, we have a factor

$$\hat{\eta}_{\alpha_1}(p_1)\, \hat{\eta}_{\alpha_2}(p_2)\, \hat{\eta}_{\alpha_3}(p_3)\, \hat{\eta}_{\alpha_4}(p_4)\, \delta(p_1 - p_2 + p_3 - p_4) \tag{83}$$

This leads to one of the following possibilities (recall that we defined $\simeq$ in Eq. (39)):

$$\begin{cases} \alpha_1 \simeq \alpha_4 \text{ and } \alpha_2 \simeq \alpha_3 \\ \qquad\quad \text{or} \\ \alpha_1 \simeq \bar{\alpha}_3 \text{ and } \alpha_3 \simeq \bar{\alpha}_3 \end{cases}$$

Thus

$$\partial_t \bar{G}_\downarrow^{(1)} = 2t(I_0 + J_0) \tag{85}$$

$$I_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int dz \langle G \eta_\alpha \eta_\beta (G - C)\, \eta_{\bar{\alpha}'}^{(-u)} \eta_{\bar{\beta}'}^{(u)} G \rangle \tag{86}$$

$$J_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int dz \langle G \eta_\alpha \eta_\beta (G - C)\, \eta_{\beta'}^{(-u)} \eta_{\alpha'}^{(u)} G \rangle \tag{87}$$

Now we can unfold the $\alpha\beta\bar{\alpha}'\bar{\beta}'$ term, cf. Eqs. (23) and (25).

$$I_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int dz \langle G \eta_\alpha \eta_{\alpha'}^{(u)} (G - C)\, \eta_{\bar{\beta}} \eta_{\bar{\beta}'}^{(u)} G \rangle \tag{88}$$

In order to decouple the $\alpha\alpha'$ and $\beta\beta'$ operators, we insert

$$1 = \int \delta(z - z')\, dz' = \int dk\, dz'\, e^{ik(z - z')} \tag{89}$$

Let us note $e^{ik\cdot}$ the operator whose kernel is

$$(e^{ik\cdot})(x, y) = e^{ikx}\, \delta(x - y) \tag{90}$$

with this notation, we have

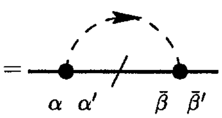
$$I_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar\beta} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int dk$$

$$\times \langle G\eta_\alpha e^{ik\cdot}\eta^{(u)}_{\alpha'}(G-C)\,\eta_{\bar\beta}\, e^{-ik\cdot}\eta^{(u)}_{\bar\beta'} G\rangle \tag{91}$$

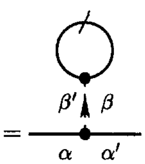
$$J_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar\beta} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int dk$$

$$\times \langle \mathrm{tr}[\eta_\beta(G-C)\,\eta^{(-u)}_{\beta'} e^{-ik\cdot}] \times G\eta_\alpha e^{ik\cdot}\eta^{(u)}_{\alpha'} G\rangle \tag{92}$$

We can adopt the following diagrammatic representation

$I_0 =$ 

(93)

$J_0 =$ 

(94)

Let us write explicitly the various incoming momenta at the half-vertices

$$I_0 = \lambda^2 \sum_{\alpha \not\simeq \beta, \bar\beta} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int d\mu(V) \int dk \int dw_1 \cdots dw_4\, dp_1 \cdots dp_4$$

$$\times G(.,w_1)^{ip_1 w_1} \hat\eta_\alpha(p_1)\, \hat\eta_{\alpha'}(p_2)\, \delta(k - p_1 + p_2)\, e^{ip_2(u-w_2)}(G-C)(w_2, w_3)$$

$$\times e^{ip_3 w_3} \hat\eta_{\bar\beta}(p_3)\, \hat\eta_{\bar\beta'}(p_4)\, \delta(p_4 - p_3 - k)\, e^{ip_4(u-w_4)} G(w_4, .) \tag{95}$$

We can see on Eq. (95) that k allows to go from the sector $\bar\beta$ to the neighboring sector $\bar\beta'$. This implies that k is restricted to a small effective domain around the origin which is a tube ζ_β whose axis is orthogonal to β and of size $O(1)\, M^{-j_1} \times O(1)\, M^{-j_0/2}$. In the same way, k goes from α to α', thus it must be in the intersection of both tubes ζ_α and ζ_β. Since the angle between those tubes is at least $M^{-rj_0/2}$, k has a norm which is at most $O(1)\, M^{rj_0/2} M^{-j_1}$.

Thus, we can freely insert a factor $\zeta_0(k)$ which restricts the integration on k to the ball of radius $O(1)\, M^{rj_0/2} M^{-j_1}$. Of course, the same analysis applies to J_0.

$$I_0 = \lambda^2 \sum_{\alpha \not\sim \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \zeta_0(k)\, dk \times \langle G\eta_\alpha e^{ik\cdot} \eta_{\alpha'}^{(u)} (G - C)\, \eta_{\bar{\beta}} e^{-ik\cdot} \eta_{\bar{\beta}'}^{(u)} G \rangle \tag{96}$$

$$J_0 = \lambda^2 \sum_{\alpha \not\sim \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \zeta_0(k)\, dk \times \langle \mathrm{tr}[\eta_\beta (G - C)\, \eta_{\beta'}^{(-u)} e^{-ik\cdot}]\, G\eta_\alpha e^{ik\cdot} \eta_{\alpha'}^{(u)} G \rangle \tag{97}$$

The point is that momentum conservation can be used in a much more efficient way. If we suppose, in Eq. (95), that the momenta $p_1,..., p_4$ are in the very low slice $\bar{\Sigma}_{j_2}$, then k will be at the intersection of two tubes of width M^{-j_2} and thus of norm less than $O(1)\, M^{rj_0/2} M^{-j_2}$.

We can introduce

$$\zeta_0 = \zeta_1 + \zeta_0(1 - \zeta_1) = \zeta_1 + (\zeta_0 - \zeta_1) \tag{98}$$

where ζ_1 forces $|k| \leqslant O(1)\, M^{rj_0/2} M^{-j_2}$. This will give two terms for I_0 (and J_0):

- a term I_1 (or J_1) with ζ_1 having a very small transfer momentum k
- a term I_2 (or J_2) with $\zeta_0(1 - \zeta_1)$ which *must* have a leg above the scale j_2.

$$I_0 = I_1 + I_2 \tag{99}$$

$$I_1 = \lambda^2 \sum_{\alpha \not\sim \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \zeta_1(k)\, dk \times \langle G\eta_\alpha e^{ik\cdot} \eta_{\alpha'}^{(u)} (G - C)\, \eta_{\bar{\beta}} e^{-ik\cdot} \eta_{\bar{\beta}'}^{(u)} G \rangle \tag{100}$$

Let $\eta_\alpha = \eta_{\alpha\uparrow\downarrow} + \eta_{\alpha\downarrow\downarrow}$, with $\eta_{\alpha\uparrow\downarrow}$ having its support above the scale j_2.

$$I_2 = \lambda^2 \sum_{(i_1,..., i_4) \neq (\downarrow\downarrow,..., \downarrow\downarrow)} \sum_{\alpha \not\sim \beta, \bar{\beta}} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int (\zeta_0 - \zeta_1)(k)\, dk \times \langle G\eta_{\alpha i_1} e^{ik\cdot} \eta_{\alpha' i_2}^{(u)} (G - C)\, \eta_{\bar{\beta} i_3} e^{-ik\cdot} \eta_{\bar{\beta}' i_4}^{(u)} G \rangle \tag{101}$$

In the following, we will forget the indices $i_1,..., i_4$ for shortness.

The terms I_2 and J_2 have a high leg, so we treat them by an analogue of Lemma 4.

Lemma 6.

$$\|I_2\|,\ \|J_2\| \leqslant O(1)(\log \lambda^{-1})\,\lambda^{1/2}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{102}$$

Proof. Again, we write the covariance as two insertions of an auxiliary field. Thanks to sector conservation, we have

$$I_2 = \sum_{|\theta| \leqslant KM^{-j_0/2}} I_3(\theta) \tag{103}$$

$$\begin{aligned} I_3(\theta) = \lambda^2 \sum_{\alpha \not\sim \beta, \bar\beta} \sum_{\gamma\delta} \int \xi(u)\, du \int dw[\hat\zeta_0(w) - \hat\zeta_1(w)] \\ \times \langle G\eta_\alpha \eta_\beta^{(w)}(G-C)\,\eta_{(\bar\gamma+\theta)}^{(-u)}\eta_{\bar\delta}^{(w+u)} G\rangle\, \delta_{\alpha\gamma} \end{aligned} \tag{104}$$

Then we write

$$\delta_{\alpha\gamma} = \langle \omega_\alpha \omega_\gamma\rangle_{d\mu_\delta(\boldsymbol{\omega})} \tag{105}$$

introducing a Gaussian random vector $\boldsymbol{\omega}$.

$$\begin{aligned} I_3(\theta) = \lambda^2 \int [\hat\zeta_0(w) - \hat\zeta_1(w)]\, dw \int d\mu(V)\, d\mu(U)\, d\mu_\delta(\boldsymbol{\omega}) \\ \times G\left(\sum_{\alpha \not\sim \beta} \eta_\alpha \omega_\alpha U \eta_\beta^{(w)}\right)(G-C)\left(\sum_{\gamma\delta} \eta_\gamma \omega_{\bar\gamma-\theta} U\eta_{\bar\delta}^{(w)}\right) G \end{aligned} \tag{106}$$

Now, we can perform a large field-small field decomposition and stairway expansions in the small field region. The leading order term corresponds to the insertion of $\lambda(\boldsymbol{\omega} * U^{j_0 j_0})$ and $\lambda^2(\boldsymbol{\omega} * U^{j_0 j_2})\, C^{j_2} V^{j_2 j_0}$. This gives a factor

$$O(1) \sup |\omega_\alpha|^2 \times \lambda^2 \times \lambda^2 M^{-(j_0-j_2)/2} \tag{107}$$

Therefore, we need to control $\langle \sup |\omega_\alpha|^2\rangle_{d\mu_\delta(\boldsymbol{\omega})}$ in order to conclude. This is done with the following lemma.

Lemma 7. Let $\boldsymbol{\omega} \in \mathbb{R}^N$ be a centered Gaussian random vector with covariance

$$\langle \omega_\alpha \omega_\beta\rangle_{d\mu_\delta(\boldsymbol{\omega})} = \delta_{\alpha\beta} \tag{108}$$

There exists a constant C_0 such that for all $N \geqslant 2$

$$\langle \sup_{\alpha} |\omega_\alpha|^2 \rangle_{d\mu_\delta(\boldsymbol{\omega})} \leqslant C_0 \log N \tag{109}$$

Proof. Using Hölder's inequality

$$\langle \sup_{\alpha} |\omega_\alpha|^2 \rangle_{d\mu_\delta(\boldsymbol{\omega})} \leqslant \langle (\sup_{\alpha} |\omega_\alpha|)^{2p} \rangle^{1/p}_{d\mu_\delta(\boldsymbol{\omega})} \langle 1^q \rangle^{1/q}_{d\mu_\delta(\boldsymbol{\omega})} \tag{110}$$

$$\leqslant \left\langle \sum_{\alpha=1}^{N} |\omega_\alpha|^{2p} \right\rangle^{1/p}_{d\mu_\delta(\boldsymbol{\omega})} \tag{111}$$

$$\leqslant N^{1/p}[(2p-1)!!]^{1/p} \leqslant 2N^{1/p}(p!)^{1/p} \tag{112}$$

Then we take $p = [\log N]$ ▮

Thus introducing the vector $\boldsymbol{\omega}$ costs only $\log \lambda^{-1}$. Then J_2 can be treated in the same way, completing the proof. ▮

We return now to the bound on I_1 and J_1.

Lemma 8.

$$\|I_1\| \leqslant O(1)(\log \lambda^{-1})\, \lambda^{1-r-4v_2} \left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{113}$$

Proof. Again we have to get rid of the constraint between α and β. We write

$$\sum_{\alpha \nsim \beta} f_{\alpha,\beta} = \sum_{\alpha\beta} f_{\alpha,\beta} - \sum_{\alpha \sim \beta} (f_{\alpha,\beta} + f_{\alpha,\bar{\beta}}) \tag{114}$$

$$= \sum_{\alpha\beta} f_{\alpha,\beta} - \sum_{|\theta| < M^{-rj_0/2}} \sum_{\alpha\beta} (f_{\alpha,\beta+\theta} + f_{\alpha,\bar{\beta}+\theta})\, \delta_{\alpha\beta} \tag{115}$$

$$= \sum_{\alpha\beta} f_{\alpha,\beta} - \sum_{|\theta| < M^{-rj_0/2}} \sum_{\alpha\beta} \langle \omega_\alpha \omega_\beta f_{\alpha,\beta+\theta} + \omega_\alpha \omega_\beta f_{\alpha,\bar{\beta}+\theta} \rangle_{d\mu_\delta(\boldsymbol{\omega})} \tag{116}$$

Putting into the expression of I_1, we get

$$I_1 = I_4 + \sum_{|\theta| < M^{-rj_0/2}} [I_5(\theta) + \bar{I}_5(\theta)] \tag{117}$$

$$I_4 = \lambda^2 \int \xi(u)\,du \int \zeta_1(k)\,dk \int d\mu(V)$$
$$\times G\left[\sum_{\substack{\alpha \\ \alpha' \simeq \alpha}} \eta_\alpha e^{ik\cdot\eta^{(u)}_{\alpha'}}\right](G-C)\left[\sum_{\substack{\beta \\ \beta' \simeq \beta}} \eta_{\bar\beta} e^{-ik\cdot\eta^{(u)}_{\bar\beta'}}\right] G \tag{118}$$

$$I_5(\theta) = \lambda^2 \int \xi(u)\,du \int \zeta_1(k)\,dk \int d\mu(V)\,d\mu_\delta(\boldsymbol{\omega})$$
$$\times G\left[\sum_{\substack{\alpha \\ \alpha' \simeq \alpha}} \omega_\alpha \eta_\alpha e^{ik\cdot\eta^{(u)}_{\alpha'}}\right](G-C)\left[\sum_{\substack{\beta \\ \beta' \simeq \beta}} \omega_\beta \eta_{\beta+\theta} e^{-ik\cdot\eta^{(u)}_{\beta'+\theta}}\right] G \tag{119}$$

$\bar I_5$ is obtained by changing β and β' into $\bar\beta$ and $\bar\beta'$ in the expression I_5.

We get small factors from the coupling constants λ^2 and the integration volume for k which is $M^{rj_0}M^{-2j_2}$. On the other hand, we must pay for the resolvents and an extra $M^{(1-r)\,j_0/2}\sup|\omega_\alpha|^2$ for $I_5 = \sum I_5(\theta)$. Hence

$$\|I_4\| \leqslant O(1)\,\lambda^{2-2r-4\nu_2}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{120}$$

$$\|I_5\| \leqslant O(1)(\log\lambda^{-1})\,\lambda^{1-r-4\nu_2}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \quad \blacksquare \tag{121}$$

Now, we are left with J_1 on which we want to apply our Ward-type identity. But we need $k.k_{\beta'}$ to be large enough. We define

$$\zeta_1(k) = \theta_{\beta'}(k) + \varepsilon_{\beta'}(k) \tag{122}$$

where $\theta_{\beta'}$ restricts the integration on k to the region $k.k_{\beta'} \geqslant \lambda^{2+\nu_3}$. This leads to

$$J_1 = J_4 + J_5 \tag{123}$$

$$J_4 = \lambda^2 \sum_{\alpha \not\simeq \beta}\ \sum_{\substack{\alpha \simeq \alpha' \\ \beta' \simeq \beta}} \int \xi(u)\,du \int \varepsilon_{\beta'}(k)\, J_{\substack{\alpha\alpha' \\ \beta\beta'}}(k)\,dk \tag{124}$$

$$J_{\substack{\alpha\alpha' \\ \beta\beta'}}(k) = \langle \operatorname{tr}[\eta_\beta(G-C)\,\eta^{(-u)}_{\beta'} e^{-ik\cdot}]\, G\eta_\alpha e^{ik\cdot\eta^{(u)}_{\alpha'}} G\rangle \tag{125}$$

Lemma 9.

$$\|J_4\| \leqslant O(1)\,\lambda^{\nu_3-\varepsilon}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{126}$$

Proof. Let us write J_4 under the following form

$$J_4 = \sum_{\beta, \beta' \simeq \beta} \int \xi(u)\, du \int d\mu(V) \times G\left[\lambda^2 \int dz\, \eta_\alpha(., z)\, T^{(u)}_{\beta\beta'}(z)\, \eta_{\alpha'}(z, .)\right] G \tag{127}$$

$$T^{(u)}_{\beta\beta'}(z) = \int \hat{\varepsilon}_{\beta'}(z - z')\, dz'(\eta^{(z')}_{\beta}, (G - C)\, \eta^{(-z'-u)}_{\beta'}) \tag{128}$$

$\eta^{(z')}_{\beta}$ and $\eta^{(-z'-u)}$ are now to be considered as functions and no longer as operators. A stairway expansion on $(G-C)$, in the small field region, proves that the leading contribution is obtained by restricting η_β and $\eta_{\beta'}$ to the very low slice j_0. This yields

$$\|T^{(u)}_{\beta\beta'}\|_\infty \leqslant O(1)\, \|\eta_{\beta_{j_0}}\|_2^2\, \|G - C\| \int |\hat{\varepsilon}_{\beta'}(x)|\, dx \tag{129}$$

$$\leqslant M^{-j_0/2} \left(\frac{\lambda^2}{\sigma}\right) \lambda^{\nu_3 - \varepsilon} \tag{130}$$

In order to get (130), we used the fact that our model is restricted to a single cube so that the integration in the direction $k_{\beta'}$, is on a domain of size $\lambda^{-2-\varepsilon}$ instead of the decaying scale $\lambda^{-2-\nu_3}$ of $\hat{\varepsilon}_{\beta'}$. The desired bound follows easily. ∎

The previous lemma would extend when we study the thermodynamic limit. In that case, when we work in a given cube Δ, V is replaced by the corresponding V_Δ whose covariance is

$$\xi_\Delta = \xi^{1/2} \chi_\Delta \xi^{1/2} \tag{131}$$

The set of all χ_Δ is a partition of unity and each χ_Δ is a smooth function with compact support around the corresponding cube Δ.[5] Then we can introduce, $\chi_{\bar{\Delta}}$ smooth with compact support around the support $\bar{\Delta}$ of χ_Δ and equal to 1 for all points whose distance to $\bar{\Delta}$ is less than $\lambda^{-2-\varepsilon}$.

Lemma 9 can be extended provided we put further localization functions at the very beginning of the expansion. In the expression of $\partial \bar{G}_\downarrow$, Eq. (75), we can replace the vertex function

$$\Gamma_\Delta(u_1, ..., u_4) = \int \xi_\Delta(x, y)\, dx\, dy\, \eta_\downarrow(u_1, x)\, \eta_\downarrow(x, u_2)\, \eta_\downarrow(u_3, y)\, \eta_\downarrow(y, u_4) \tag{132}$$

by the following one

$$\tilde{\Gamma}_{\Delta}(u_1,..., u_4) = \int \xi_{\Delta}(x, y)\, dx\, dy (\chi_{\bar{\Delta}}\eta_{\downarrow})(u_1, x)(\eta_{\downarrow}\chi_{\bar{\Delta}})(x, u_2)$$

$$\times (\chi_{\bar{\Delta}}\eta_{\downarrow})(u_3, y)(\eta_{\downarrow}\chi_{\bar{\Delta}})(y, u_4) \tag{133}$$

The error term is very small because of the fast decay of $\eta_{\downarrow}$ on a scale $M^{j_1} \ll \lambda^{-2-\varepsilon}$ and the functions $\chi_{\bar{\Delta}}$ force the tadpole to stay in a cube close to Δ.

5.6. Ward Term

Finally, we must deal with

$$J_5 = \lambda^2 \sum_{\alpha \not\simeq \beta} \sum_{\substack{\alpha \simeq \alpha' \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \theta_{\beta'}(k)\, J_{\substack{\alpha\alpha' \\ \beta\beta'}}(k)\, dk \tag{134}$$

We set

$$C_t = \frac{1}{p^2 - E - i\sigma - (1-t^2)\,\Sigma} \tag{135}$$

Our Ward-type identity relies on the identity

$$2k.k_{\beta'} = (p+k)^2 - p^2 - 2k.(p-k_{\beta'}) - k^2 \tag{136}$$

$$= C_t(p+k)^{-1} - C_t(p)^{-1} - 2k.(p-k_{\beta'}) - k^2$$

$$-(1-t^2)[\Sigma(p+k) - \Sigma(p)] \tag{137}$$

In momentum space, the tadpole insertion of $J_{\substack{\alpha\alpha' \\ \beta\beta'}}$ can be written

$$\mathscr{T}(k) = \mathrm{tr}[\eta_{\beta'}^{(-u)} e^{-ik\cdot}\eta_{\beta}(G-C)]$$

$$= \int dp\, \eta_{\beta'}(p)\, e^{-ipu}\eta_{\beta}(p+k)(G-C)(p+k, p) \tag{138}$$

We insert Eq. (137) to get

$$\begin{aligned}(2k.k_{\beta'})\,\mathcal{T}(k) = {} & \mathrm{tr}[\eta_{\beta'}^{(-u)} e^{-ik\cdot}\eta_\beta C_t^{-1}(G-C)] \\ & - \mathrm{tr}[\eta_{\beta'}^{(-u)} e^{-ik\cdot}\eta_\beta (G-C)\, C_t^{-1}] \\ & - 2k.\mathrm{tr}[\eta_\beta(G-C)\, D_{\beta'}^{(-u)} e^{-ik\cdot}] - 2^2\mathcal{T}(k) \\ & + (1-t^2)\,\mathrm{tr}[\eta_\beta(G-C)\,\eta_{\beta'}^{(-u)}(\Sigma e^{-ik\cdot} - e^{-ik\cdot}\Sigma)]\end{aligned} \tag{139}$$

$$D_{\beta'}(x,y) = \int dp\, e^{ip(x-y)}(p-k_{\beta'})\,\eta_{\beta'}(p) \tag{140}$$

Now, we use the resolvent identity

$$G = C_t - C_t\,\lambda VG = C_t - G\,\lambda VC_t \tag{141}$$

Since C and C_t commute, we get

$$\begin{aligned}(2k.k_{\beta'})\,\mathcal{T}(k) = {} & -\mathrm{tr}[\eta_{\beta'}^{(-u)} e^{-ik\cdot}\eta_\beta\,\lambda VG] + \mathrm{tr}[G\,\lambda V\eta_{\beta'}^{(-u)} e^{-ik\cdot}\eta_\beta] \\ & - 2k.\mathrm{tr}[\eta_\beta(G-C)\, D_{\beta'}^{(-u)} e^{-ik\cdot}] - k^2\mathcal{T}(k) \\ & + (1-t^2)\,\mathrm{tr}[\eta_\beta(G-C)\,\eta_{\beta'}^{(-u)}(\Sigma e^{-ik\cdot} - e^{-ik\cdot}\Sigma)]\end{aligned} \tag{142}$$

We put (142) back into the expression of J_5, writing

$$J_5 = -J_R + J_L - J_D - J_{k^2} + J_\Sigma \tag{143}$$

where the notations refer directly to the various terms of Eq. (142).

Lemma 10.

$$\|J_D\| \leqslant O(1)(\log\lambda^{-1})\,\lambda^{1-r-2v_2-\varepsilon}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{144}$$

$$\|J_{k^2}\| \leqslant O(1)(\log\lambda^{-1})\,\lambda^{2-2r-4v_2-\varepsilon}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{145}$$

$$\|J_\Sigma\| \leqslant O(1)(\log\lambda^{-1})\,\lambda^{2-r-2v_2-\varepsilon}\left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{146}$$

Proof. We can treat J_D, J_{k^2} and J_Σ the way we treated J_4 because we have earned small factors:

- for J_D, we earn something thanks to

$$k.D_{\beta'} \sim |K| \, M^{-j_0/2} \eta_{\beta'} \sim M^{-(1-r)\, j_0/2} M^{-j_2} \tag{147}$$

but we have still the spatial integration of the tadpole to pay, which costs

$$\int dx \left| \widehat{\left(\frac{\theta_{\beta'}}{2k.k_{\beta'}} \right)}(x) \right| \leqslant O(1)(\log \lambda^{-1}) \, \lambda^{-2-\varepsilon} \tag{148}$$

This is because we have

$$\left\| \widehat{\frac{\theta_{\beta'}}{2k.k_{\beta'}}} \right\|_\infty \leqslant O(1) \, M^{r(j_0/2)} M^{-j_2} \int_{\lambda^{2+\nu_3}}^{M^{r(j_0/2)} M^{-j_2}} \frac{dk_{//}}{2 \, |k_{\beta'}| \, k_{//}} \tag{149}$$

$$\leqslant O(1) \, M^{r(j_0/2)} M^{-j_2} \log \lambda^{-1} \tag{150}$$

and the spatial integration is in a volume $O(1) \, M^{r(j_0/2)} M^{+j_2} \times \lambda^{-2-\varepsilon}$. Therefore

$$\|J_D\| \leqslant O(1)(\log \lambda^{-1}) \, \lambda^{1-r-2\nu_2} \left(\frac{\lambda^2}{\sigma} \right)^2 \times \frac{1}{\sigma} \tag{151}$$

- for J_{k^2}, we earn something from

$$|k^2| \leqslant M^{rj_0} M^{-2j_2} \tag{152}$$

and the spatial integration of the tadpole has the same price as before.

$$\|J_D\| \leqslant O(1)(\log \lambda^{-1}) \, \lambda^{2-2r-4\nu_2-\varepsilon} \left(\frac{\lambda^2}{\sigma} \right)^2 \times \frac{1}{\sigma} \tag{153}$$

- finally, for J_Σ, we notice that Σ is an almost local operator whose norm is proportional to λ^2. Thus, taking the commutator with $e^{-ik\cdot}$ gives a gradient term which is very small

$$\|[\Sigma, e^{-ik\cdot}]\| \leqslant O(1) \, \lambda^2 \, |k| \tag{154}$$

We can conclude

$$\|J_\Sigma\| \leqslant O(1)(\log \lambda^{-1}) \, \lambda^{2-r-2\nu_2-\varepsilon} \left(\frac{\lambda^2}{\sigma} \right)^2 \times \frac{1}{\sigma} \quad \blacksquare \tag{155}$$

We are left with $-J_R + J_L$

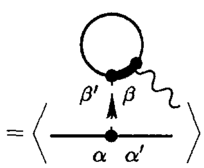
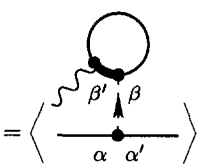
$$-J_R = -\lambda^2 \sum_{\alpha \not\sim \beta, \beta} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \frac{\theta_{\beta'}(k)}{2k.k_{\beta'}}\, dk$$

$$\times \langle \mathrm{tr}[\eta_{\beta'}^{(-u)} e^{-ik\cdot} \eta_\beta \lambda V G]\, G\eta_\alpha e^{ik\cdot} \eta_{\alpha'}^{(u)} \rangle \tag{156}$$

$$J_L = \lambda^2 \sum_{\alpha \not\sim \beta, \bar\beta} \sum_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \int \xi(u)\, du \int \frac{\theta_{\beta'}(k)}{2k.k_{\beta'}}\, dk$$

$$\times \langle \mathrm{tr}[G \lambda V \eta_{\beta'}^{(-u)} e^{-ik\cdot} \eta_\beta]\, G\eta_\alpha e^{ik\cdot} \eta_{\alpha'}^{(u)} \rangle \tag{157}$$

At this point, we use a diagrammatic representation to perform the connection with the heuristic presentation of Section 4. It is easy to see that we have

$J_R =$  and $J_L =$  (158)

We take the degenerate part of the V away

$$\eta_\beta V = \sum_{\gamma \sim \beta, \bar\beta} \eta_\beta V \eta_\gamma + \sum_{\gamma \not\sim \beta, \bar\beta} \eta_\beta V \eta_\gamma \tag{159}$$

so that we can write

$$J_{R,L} = J_{R,L}^{(0)} + J_{R,L}^{(1)} \tag{160}$$

$J_{R,L}^{(0)}$ is the almost diagonal V part, it has a bound

$$\| J_{R,L}^{(0)} \| \leqslant O(1)(\log \lambda^{-1})\, \lambda^{r/2-\varepsilon} \left(\frac{\lambda^2}{\sigma}\right)^2 \times \frac{1}{\sigma} \tag{161}$$

We integrate the V by parts in $J_{R,L}^{(1)}$, and we use sector conservation and unfolding to generate the 12 terms of Eq. (36).

Lemma 11. There exists $\nu > 0$ such that

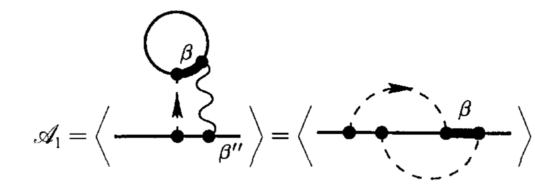
$$\| J_{R,L}^{(1)} \| \leqslant O(1)(\log \lambda^{-1})^3\, \lambda^{\nu} \left(\frac{\lambda^2}{\sigma}\right)^3 \times \frac{1}{\sigma} \tag{162}$$

where

$$\nu = \min\left\{(2-r-4\nu_1-2\nu_2), \left(\nu_2+\frac{r}{2}-\varepsilon\right),\right.$$
$$\left.(1-r-2\nu_2-\nu_3-\varepsilon), 2(\nu_1-\nu_2-r)-\varepsilon\right\} \tag{163}$$

Proof. Let us bound the various terms of (36). First, we consider the graphs without loops. We explain the bound for a typical one

$\mathscr{A}_1 =$ (164)



One can check that the analytic expression for $\mathscr{A}_1$ is

$$\mathscr{A}_1 = \lambda^4 \sum_{\substack{\alpha\not\simeq\beta,\bar\beta\\ \gamma\not\simeq\beta,\bar\beta}} \sum_{\substack{\alpha'\simeq\alpha\\ \beta'\simeq\beta}} \sum_{\substack{\beta''\simeq\beta\\ \gamma'\simeq\gamma}} \int \xi(u)\,\xi(v)\,du\,dv \int \frac{\theta_{\beta'}(k)}{2k.k_{\beta'}}\,\zeta_0(k')\,dk\,dk'$$
$$\times \langle G\eta_\alpha e^{ik\cdot\eta_{\alpha'}^{(u)}} G\eta_{\gamma'}^{(-v)} e^{ik'\cdot\eta_\gamma} G\eta_{\beta'}^{(-u)} e^{-ik\cdot\eta_\beta} e^{-ik'\cdot\eta_{\beta''}^{(v)}} G\rangle \tag{165}$$

We bound $\mathscr{A}_1$ the way we bounded I_1 in Lemma 8.

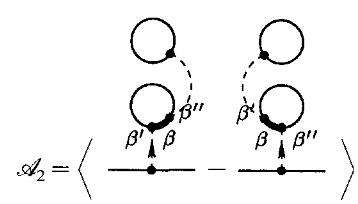
- Our small factors are λ^4 and the integration on k and k', i.e., $(\log\lambda^{-1})\,M^{rj_0/2}M^{-j_2}$ and $M^{rj_0}M^{-2j_1}$.
- we must get rid of the constraints on α, β, and γ. This is done by introducing Gaussian random vectors and costs $(M^{(1-r)j_0/2}\log\lambda^{-1})^2$, Finally we must pay for the resolvents.

Gathering all factors, we get

$$\|\mathscr{A}_1\| \leqslant O(1)(\log\lambda^{-1})^3\,\lambda^{2-r-4\nu_1-2\nu_2}\left(\frac{\lambda^2}{\sigma}\right)^3 \times \frac{1}{\sigma} \tag{166}$$

Now, let us see how we can pair the graphs with loops to get a small result. For instance let us consider

$\mathscr{A}_2 =$ (167)



We know that the momentum k at the first vertex is bounded by $K_1 M^{rj_0/2} M^{-j_2}$. Therefore, we can find K_2 such that if the momentum k' at the second vertex is larger than $K_2 M^{rj_0} M^{-j_2}$ in norm, we have a leg higher than $2K_1 M^{rj_0/2}$. This leads to the decomposition

$$\mathscr{A}_2 = \mathscr{A}_2^{\text{high}} + \mathscr{A}_2^{\text{low}} \tag{168}$$

In $\mathscr{A}_2^{\text{high}}$, we have a high leg at the second vertex. But this leg can be η_β (the thick line) and this would prevent us from making a stairway expansion and getting a small factor. Yet, in that case, we would know that at the first vertex, $\eta_{\beta'}$ (resp. $\eta_{\beta''}$) had to be higher than $K_1 M^{rj_0/2} M^{-j_2}$. Therefore, in the same way we did in Lemma 10 we can show

$$\|\mathscr{A}_2^{\text{high}}\| \leqslant O(1)(\log \lambda^{-1})\, \lambda^{\nu_2 + r/2 - \varepsilon} \left(\frac{\lambda^2}{\sigma}\right)^3 \times \frac{1}{\sigma} \tag{169}$$

For $\mathscr{A}_2^{\text{low}}$, we use the fact that the first graph is equal to the second up to error terms.

- We change $\theta_{\beta'}(k)/2k.k_{\beta'}$ into $\theta_\beta(k)/2k.k_\beta$. The remainder term bears a factor

$$|k.(k_\beta - k_{\beta'})|\, \lambda^{-2-\nu_3} \leqslant O(1)\, \lambda^{1-r-2\nu_2-\nu_3} \tag{170}$$

- We exchange the ends of the two dashed lines in the middle loop. This amounts to commute first $e^{-ik\cdot}$ and η_β and then $e^{-ik'}$ and η_β. Thus the error term has an extra factor

$$\max(|k|, |k'|)\, M^{j_1} \leqslant O(1)\, M^{rj_0} M^{-j_2} M^{j_1} \leqslant O(1)\, \lambda^{2(\nu_1 - \nu_2 - r)} \tag{171}$$

In conclusion, we obtain

$$\|\mathscr{A}_2^{\text{low}}\| \leqslant O(1)(\log \lambda^{-1}) \max[\lambda^{1-r-2\nu_2-\nu_3-\varepsilon}, \lambda^{2(\nu_1-\nu_2-r)-\varepsilon}] \left(\frac{\lambda^2}{\sigma}\right)^3 \times \frac{1}{\sigma} \tag{172}$$

Taking ν_1 small (but not too small) and r, ν_2 and ν_3 very small, the various powers of λ (standing for the small factors we earned) that we met all along the demonstration are indeed positive. This concludes the Proof of Theorem 1.

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